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On the Heterogeneous Vehicle Routing Problem under Demand Uncertainty

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Abstract In this paper we study the heterogeneous vehicle routing problem under demand uncertainty, on which there has been little research to our knowledge. The focus of the paper is to provide a strong formulation that also easily allows tractable robust and chance-constrained counterparts. To this end, we propose a basic Miller-Tucker-Zemlin (MTZ) formulation with the main advantage that uncertainty is restricted to the right-hand side of the constraints. This leads to compact and tractable counterparts of demand uncertainty. On the other hand, since the MTZ formulation is well known to provide a rather weak linear programming relaxation, we propose to strengthen the initial formulation with valid inequalities and lifting techniques and, furthermore, to dynamically add cutting planes that successively reduce the polyhedral region using a branch-and-cut algorithm. We complete our study with extensive computational analysis with different performance measures on different classes of instances taken from the literature. In addition, using simulation, we conduct a scenario-based risk level analysis for both cases where either unmet demand is allowed or not.

Keywords stochastic vehicle routing problem · robust optimization · chance constrained programming

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1 Introduction

The Capacitated Vehicle Routing Problem (CVRP), with its many variants, is one of the most widely studied NP-hard problems in combinatorial optimization due to its many practical applications and theoretical challenges. The classical CVRP is defined on an arc weighted directed graph $G = (V, A)$ with routing costs $c_a, a \in A$. It consists in serving a set of customers $V_c = \{1, \dots, n\}$ with known demand $q_i,\ i \in V_c$, using a fleet of vehicles with identical capacity Q and located at the same (unique) depot (usually denoted as 0 in the graph, i.e., $V = \{0\} \cup V_c$). Each vehicle takes exactly one route starting from the depot, visiting a subset of the customers and returning to the depot. Customer demand cannot be split among different routes and the sum of demands in each route must not exceed the vehicle capacity Q. The solution of the CVRP is a minimum cost partition of the customers according to the vehicle routes.

There is a broad literature on heuristic algorithms for the CVRP, but there are much fewer exact methods available, especially for its more complex variants. In this paper, we consider an important generalization of the classical CVRP known as *Heterogeneous Vehicle Routing Problem* (HVRP), in which a heterogeneous fleet of vehicles is stationed at the depot and is used to serve the customers. There are m different vehicle types: $K = \{1, \dots, m\}$. For each type $k \in K$, U_k vehicles are available at the depot with capacity Q_k, where $Q_1 < \dots < Q_m$. Each vehicle type k can also be associated with a fixed cost F_k and the routing costs can be vehicle dependent $c^k_a,\ a \in A,\ k \in K$. This is usually called *Heterogeneous VRP with Fixed Costs and Vehicle Dependant Routing Costs* (HVRPFD). A strongly related problem that has received much attention in the literature is the *Multi-Depot* VRP (MDVRP), characterized by a fleet of unlimited identical vehicles of capacity Q, located at p depots. Any MDVRP instance can be converted into an equivalent HVRP instance. Finally, variants with an unlimited number of vehicles are called *Fleet Size and Mix* (FSM).

The focus of this paper is to study the HVRP when customer demands are *uncertain*. There are many ways to deal with uncertainty. Here we consider three uncertainty frameworks: two *robust* counterparts of Ben-Tal & Nemirovski [4] and Bertsimas & Sim [5], and a *chance-constrained* counterpart (see Charnes & Cooper [8],[9]).

To achieve this, our first step is to formulate the deterministic problem in such a way that the corresponding counterparts of uncertainty remain tractable via mixed integer linear programming (MILP). We propose a basic Miller-Tucker-Zemlin (MTZ) [21] formulation, the main advantage being that uncertainty is restricted to the right-hand side of the constraints. This leads to compact and tractable uncertain counterparts. Since the MTZ formulation is well known to provide a rather weak linear programming (LP) relaxation, which performs poorly when plugged into a branch-and-bound framework, our work in the following steps aim to overcome this weakness. Exact solution methods employ two general strategies to improve the approximation of the convex hull of an MILP problem. The first strategy is "static" and tries to tighten the polyhedral representation of the initial formulation before any computational solution procedure is started, while the second strategy is "dynamic" and keeps adding cutting planes during run-time, which successively reduces the size of the polyhedral region. The second step of our work is to integrate both strategies using lifting techniques and cutting-planes within a branch-and-cut algorithm.

A solution obtained from the above approaches is known as *pre-planned routes* and does not consider failure cost. In order to have a realistic picture of a vehicle routing problem, we perform an extensive computational analysis. We first compare deterministic, robust and chance-constrained solutions based on three performance measures: (i) the extra cost required for achieving a certain level of validity for routes of the deterministic solution, (ii) the unmet demand and the number of unmet customers whom the vehicles fail to serve on their planned routes, (iii) the recourse cost, which is the extra cost, in case of failure, of returning to the depots for replenishment and resuming the route. Moreover, using a scenario-based analysis, we analyze and search for the best risk level at which the total of the pre-planned route cost and the recourse, or lost sale, cost is minimized.

In this paper, we study the HVRP with unlimited number of vehicles and the multi-depot HVRP with limited number of vehicles. The structure of the paper is as follows. Section 2 is devoted to a brief literature review. In Section 3, we present our basic MTZ deterministic model followed by valid inequalities along with lifting techniques to strengthen the initial formulation, i.e., at the root node of the branch-and-bound tree. In Section 4, the uncertainty counterparts are presented for the three aforementioned frameworks. Also, new probability bounds are proposed to calculate the parameters of the Bertsimas & Sim robust approach. In Section 5, we present extensive computational results using different classes of instances taken from the literature. We complete our study with some concluding remarks in Section 6.

2 Literature Review

The first study on the HVRP is by Golden *et al.* [16], which presents various lower bounds. Yaman [29] moves forward and shows six different formulations, derives valid inequalities and lifting techniques. Apparently, the most effective algorithms are based on a set-partitioning formulation and exploit advanced column-generation techniques. In particular, Baldacci & Mingozzi [3] present a first unified framework based on a set-partitioning formulation for solving HVRP and some variants that can be seen as special cases. The framework is extended in Baldacci *et al.* [1] to include other variants. Finally, Baldacci *et al.* [2] present new valid inequalities for a two-commodity flow HVRP formulation.

The main consequence of uncertain demand is that a planned vehicle route may exceed the vehicle capacity. In such a case, *failure* is said to occur. There are two main approaches to dealing with uncertainty: stochastic optimization and robust optimization.

Stochastic optimization models take advantage of the fact that probability distributions on the data are known or can be estimated. The goal is to find a solution that maximizes (or minimizes) the expectation of some function of the decision and the random variables. There are several studies on stochastic CVRP (SVRP) in the literature. The most recent surveys are Gendreau *et al.* [15], Dror [12] and Erera *et al.* [14]. The first result on SVRP dates back to the late 1960s with Tillman [28]. In the 1980s SVRP received more attention with Stewart & Golden [26], Dror & Trudeau [13], Laporte & Louveau [17] and Laporte *et al.* [18]. We distinguish two main stochastic optimization models: two-stage recourse and chance-constrained. The *two-stage recourse* models, in case of failure, implement a *recourse action* (generating extra cost). There are two different solution concepts within the two-stage recourse model: *a priori optimization* as described by Bertsimas [6], and the *re-optimization strategy* (see Secomandi & Margot [23]). On the one hand, re-optimization gives better results in terms of solution quality. On the other hand, a priori optimization is preferable from a computational point of view since it entails solving only one instance of VRP. Algorithmically, the two-stage recourse strategy can be either tackled using heuristics or branch-and-cut methods based on the integer L-Shaped method by Laporte & Louveaux [19]. Alternatively, Novoa *et al.* [22] and Christiansen & Lysgaard [10] propose a set-partitioning formulation and use column generation to solve it. In the *chance-constrained* models failure can happen within some (small) *probability bound*. Stewart & Golden [26], Laporte *et al.* [18] showed that chance constrained counterparts are equivalent to the deterministic VRP for a number of routing problems and uncertainty assumptions.

If we have no knowledge on the data, one approach to tackling such problems is called *robust optimization*. Here the goal is to find routes that are feasible for all demand (scenario) realizations, so failure can *never occur*. Literature is rather scarce on this topic and we are only aware of a recent study by Sungur *et al.* [27], who use the robust optimization methodology introduced by Ben-Tal & Nemirovski [4] to formulate the Robust CVRP (RVRP).

To our knowledge there is no literature on HVRP under uncertainty. It is important to point out that our aim in this study is to compare *robust* and *chance-constrained* HVRP models *under demand uncertainty*. So the main focus is not on effectively solving the deterministic problem, but on formulating it in such a way that the uncertainty counterparts remain tractable. For this reason, we resort to a Miller-Tucker-Zemlin formulation [21] that, although providing a rather weak LP relaxation, allows us to represent the uncertainty counterparts as tractable MILP models. In particular, we consider three uncertainty frameworks: two *robust* counterparts according to Ben-Tal & Nemirovski [4] and Bertsimas & Sim [5], and a chance-constrained counterpart (see Charnes & Cooper [8],[9]).

3 Deterministic Model

In this section we present an MTZ formulation for the (deterministic) HVRP and some techniques to strengthen the model.

3.1 Formulation

The HVRP can be formally defined as follows. We are given a complete directed graph $G = (V, A)$, where $V = \{0, \ldots, n\}$ is the set of vertices, A the set of arcs and $A_c \subset A$ is the subset of arcs between customers. Node 0 denotes the (unique) depot and the other vertices $V_c = \{1, \ldots, n\}$ represent customers. A fleet of heterogeneous vehicles is stationed at the depot. Without loss of generality we assume that there are m different vehicle types $K = \{1, \ldots, m\}$ and, for each type $k \in K$, there is only one vehicle available with capacity $Q_k > 0$, where $Q_1 < \cdots < Q_m$. Accordingly K corresponds to the set of all vehicles and m is the total number of vehicles available at the depot. The cost of traveling from node i to node j (arc $a = (i, j)$) by vehicle k is denoted by c_a^k. Each customer i has an integer demand q_i, with $0 < q_i \leq Q_m$. Each customer must be served by exactly one vehicle, so demand cannot be split. No vehicle can serve a set of customers whose total demand exceeds its capacity. The problem is to find m vehicle routes of minimum cost, where each vehicle leaves the depot, visits a subset of customers and finally returns to the depot.

There are three main classes of formulations: *vehicle flow*, *two-commodity flow* and *set partitioning*. Among the vehicle flow formulations, we distinguish the *two-index* vehicle flow formulation, which uses x_{ij}, $a = (i, j) \in A$ variables, and the *three-index* vehicle flow formulation, which uses x_{ij}^k, $a = (i, j) \in A$, $k \in K$ variables. We will use the later formulation as it is particularly suited for heterogeneous vehicles.

Let x_a^k be a binary variable, indicating whether vehicle k travels from node i to node j (arc $a = (i, j)$). Also, let u_i, $i \in V_c$, be a continuous variable representing the total demand of nodes on the route till (customer) node i (including node i). Finally, given a node $i \in V$, let $\delta^-(i)$ and $\delta^+(i)$ denote the set of incoming and outgoing arcs, respectively, of node i ($\delta(i) = \delta^+(i) \cup \delta^-(i)$). The MILP formulation is then:

$$\min \quad \sum_{k\in K} \sum_{a\in A} c_a^k x_a^k \tag{1}$$

$$\text{s.t.} \quad \sum_{a\in\delta^+(i)} x_a^k - \sum_{a\in\delta^-(i)} x_a^k = 0, \quad i \in V,\ k \in K \tag{2}$$

$$\sum_{k\in K} \sum_{a\in\delta^+(i)} x_a^k = 1, \quad i \in V_c \tag{3}$$

$$\sum_{k\in K} \sum_{a\in\delta^-(i)} x_a^k = 1, \quad i \in V_c \tag{4}$$

$$\sum_{a\in\delta^+(0)} x_a^k = 1, \quad k \in K \tag{5}$$

$$\sum_{a\in\delta^-(0)} x_a^k = 1, \quad k \in K \tag{6}$$

$$-u_j + u_i + Q_m \sum_{k\in K} x_a^k \leq Q_m - q_j, \quad a = (i, j) \in A_c \tag{7}$$

$$q_i \le u_i \le \sum_{k\in K} Q_k \sum_{a\in\delta^+(i)} x_a^k, \quad i \in V_c \tag{8}$$

$$x_a^k \in \{0,1\}, \quad a \in A, \ k \in K. \tag{9}$$

The degree equations (2–6) ensure that all customers are visited exactly once and for each vehicle there is exactly one route starting from the depot and returning to the depot. Inequalities (7–8) are known as Miller-Tucker-Zemlin (MTZ) constraints. They ensure that the routes are connected and, at the same time, impose vehicle capacity restrictions. Constraints 9 are the integrality conditions on the x_a^k variables.

Let $V_d = \{n+1, .., N\}$ be the set of depots. In the above model, to obtain MDHVRP, we can replace V with $V = V_c \cup V_d$ and accordingly A will be updated to the set of arcs connecting the nodes in V.

3.2 Valid Inequalities

We present two well-known types of valid inequalities for the CVRP, which can be easily extended to the HVRP.

3.2.1 Capacity Inequalities

The first type of inequalities forbids any route exceeding the vehicle capacity. Note that the current MTZ constraints (7–8) already forbid such routes. The only reason for introducing these inequalities is to strengthen the LP relaxation, as also mentioned by Yaman [29]:

$$\sum_{i\in V_c} \sum_{a\in\delta^+(i)} q_i x_a^k \le Q_k, \ k \in K. \tag{10}$$

3.2.2 Subtour Elimination Inequalities

It is well known that any valid inequality for the two-index vehicle flow formulation can be transformed into a valid inequality for the three-index vehicle flow formulation by using $x_a = \sum_{k=1}^m x_a^k$. These inequalities are called *aggregated* by Letchford & Salazar-González [20]. *Subtour elimination* inequalities are rather common constraints for the CVRP two-index vehicle flow formulation, sometimes called *rounded capacity* inequalities. They forbid subtours and routes that exceed the vehicle capacity by imposing, for any subset S of customers that does not include the depot, that at least $\lceil q(S)/Q \rceil$ vehicles enter and leave S, where $q(S) = \sum_{i\in S} q_i$ and Q is the vehicle capacity. Here we present an extension to the three-index vehicle flow representation for the heterogeneous case. Let $(S : T) = \{(i,j) \in A : i \in S, j \in T\}$ and $X(S : T) = \sum_{k\in K} \sum_{(i,j)\in(S:T)} x_{ij}^k$. For any $S \subseteq V_c$, the inequality

$$X(S : \bar{S}) \ge \left\lceil \frac{q(S)}{Q_m} \right\rceil \tag{11}$$

is a valid inequality for the HVRP three-index vehicle flow formulation ($\bar{S} = V_c \backslash S$). Note that, although this extension provides valid inequalities for HVRP and forbids all subtours, it may allow routes that exceed the vehicle capacity. This is due to the fact that in the HVRP the right-hand side of the inequality depends on the capacity of the vehicle (and hence, by using Q_m, we overestimate the denominator), whereas in the classical CVRP, all vehicles have the same capacity Q. To overcome this problem we use Yaman [29] and disaggregate such inequalities in the following way:

$$X(S : \bar{S}) \geq \left\lceil \frac{q(S)}{Q_k} \right\rceil, \; k \in K, \; S \subseteq V_c. \tag{12}$$

3.3 Lifting Technique

It is known that valid inequalities can be strengthened via lifting. Desrochers & Laporte [11] propose a simple lifting technique for the MTZ constraints for the TSP. Here we extend their technique to the HVRP. To simplify notation we denote by $x_{ij} = \sum_{k \in K} x_{ij}^k$.

Proposition 1 *The lifted version of constraints (7) is as follows:*

$$-u_j + u_i + Q_m x_{ij} + (Q_m - q_j - q_i) x_{ji} \leq Q_m - q_j, \; (i,j) \in A_c. \tag{13}$$

Proof. If $x_{ij} = 1$ then $x_{ji} = 0$, so we obtain the original MTZ inequality. On the other hand, if $x_{ji} = 1$, then the inequality reduces to $u_i \leq u_j + q_i$, which is again valid according to MTZ. □

Similarly it is possible to lift the MTZ upper bound in (8) as follows:

$$u_i \leq \sum_{k \in M} Q_k \sum_{j \in V} x_{ij}^k - \sum_{j \in V_c} q_j x_{ij}, \; i \in V_c. \tag{14}$$

For any customer $i \in V_c$, its successor can be either another customer or a depot. If it is a customer $j \in V_c$, then $u_i \leq u_j - q_j$ is valid. If it is a depot, the term $\sum_{j \in V_c} q_j x_{ij}$ is zero and we obtain the original MTZ upper bound. We call the model of (1–6) & (8–9) & (13-14) HVRP-DL for brevity.

3.4 Reformulation and Linearization Technique

We apply a specialized version of the well-known Reformulation-Linearization Technique (RLT) by Sherali & Adams [24]. In particular, to contain the size of the resulting model, we follow Sherali & Driscoll [25], who only apply a partial first-level RLT version and provide a relatively tight formulation for the TSP.

We start by restating the MTZ constraints (7) as follows:

$$u_j x_{ij} = (u_i + q_j) x_{ij}, \; (i,j) \in A_c \tag{15a}$$

$$u_j x_{0j} = q_j x_{0j}, \; j \in V_c \tag{15b}$$

We call the model (1–6) & (8–9) & (15a–15b) HVRP-NL for brevity.

We now apply the specialized version of RLT by Sherali & Driscoll [25] to HVRP-NL. The approach consists of two steps. First, we *reformulate* by generating additional (nonlinear) implied constraints. Second, we *linearize* the nonlinear terms using a substitution of variables in place of each distinct nonlinear term.

Reformulation: We reformulate the HVRP-NL by generating three sets of quadratic constraints as follows.

(S1): Multiply by u_i both of the degree constraints (3) and (4).
(S2): Multiply the first inequalities in (8) by x_{ij} and $(1-x_{ij}-x_{ji})$, respectively.
(S3): The second inequalities in (8) suggest that $(Q_m - u_j) \geq 0$, which we multiply by x_{ij} and $(1 - x_{ij} - x_{ji})$, respectively.

Linearization: We linearize the HVRP-NL along with the three new sets of constraints (S1)–(S3) generated above using the following substitution of variables:

$$y_{ij} = u_i x_{ij} \text{ and } z_{ij} = u_j x_{ij}. \tag{16}$$

Note that y_{ij} can be interpreted as the load of the vehicle *before* visiting customer j, if j is served after customer i, zero otherwise. Similarly, z_{ij} can be interpreted as the load of the vehicle *after* visiting customer j, if j is served after customer i, zero otherwise. Also, we can replace $u_j x_{0j}$ by $q_j x_{0j}$ using (15b), and we can bound $u_j x_{j0}$ from above using $Q_k x_{j0}$. Note that we can always eliminate z_{ij} using the relationship $z_{ij} = y_{ij} + q_j x_{ij}$. The linearization step yields the inequalities given below.

Proposition 2 *Denote by $\delta_c^+(i)$ the set of arcs $(i,j) \in A_c$. Linearization of (S1) leads to the following:*

$$\sum_{(i,j)\in\delta_c^+(i)} y_{ij} + \sum_{k\in K} Q_k x_{i0}^k - u_i \geq 0. \tag{17}$$

and

$$\sum_{(j,i)\in\delta_c^-(i)} z_{ji} + q_i x_{0i} - u_i = 0. \tag{18}$$

Proof. Multiplying (3) by u_i we obtain

$$\sum_{(i,j)\in\delta^+(i)} u_i x_{ij} - u_i = 0.$$

Then substituting y_{ij} and observing that the load of a vehicle u_i leaving customer i and entering the depot must be less than or equal the capacity of the vehicle Q_k, yields the inequalities. Similarly, multiplying (4) by u_i we obtain

$$\sum_{(j,i)\in\delta^-(i)} u_i x_{ji} - u_i = 0.$$

Then substituting z_{ji} and using (15b) we obtain the equations. □

Next, (S2) and (S3) can be linearized simply by substituting the quadratic terms with their corresponding variables. Hence, linearization of (S2) leads to

$$z_{ij} \geq q_j x_{ij}, \tag{19a}$$
$$u_j \geq z_{ij} + y_{ji} + q_j - q_j x_{ij} - q_j x_{ji}; \tag{19b}$$

and linearization of (S3) leads to:

$$z_{ij} \leq Q_m x_{ij}, \tag{20a}$$
$$u_j \leq Q_m(1 - x_{ij} - x_{ji}) + z_{ij} + y_{ji}. \tag{20b}$$

Note that in all the new sets of constraints introduced above, z_{ij} can be eliminated with substitution of $y_{ij} + q_j x_{ij}$.

Extending the argument of Sherali & Driscoll [25], we conclude on validity and the tightness of our new formulation as follows.

Proposition 3 *The formulation obtained by replacing (7–8) with (17), (18), (19a–20b) is valid and provides an LP relaxation that is tighter than the LP relaxation of the HVRP-DL.*

Proof. The validity follows by construction. Hence it suffices to show that the constraints (17), (18), (19a–20b) imply (13). To do so, first we replace z_{ij} with $y_{ij} + q_j x_{ij}$ in (19b) and in (20b), then we multiply (20b) by -1 and finally we interchange i and j in (20b). By surrogating the resulting inequalities we obtain

$$0 \geq u_i - u_j - Q_m + (Q_m - q_i - q_j)x_{ji} + Q_m x_{ij} + q_j,$$

which is (13). □

This proposition will be supported by computational experiments in Section 5.

4 Models of Demand Uncertainty

Now we are ready to move to the models we are interested in, i.e., when customer demands q are subject to uncertainty. We present two robust counterparts of Bent-Tal & Nemirovski and of Bertsimas & Sim, and a chance-constrained counterpart.

4.1 Ben-Tal & Nemirovski Robust Model

In Ben-Tal & Nemirovski (BN) model, the uncertain demand vector q belongs to a bounded uncertainty set U, which is constructed as a set of deviations around a fixed *expected value* q^0. In the following, we let s denote the number of *(demand) scenario vectors*: $q^1, \ldots, q^s$. The uncertainty set U consists of linear combinations of the scenario vectors with *weights* $\xi \in \Xi$:

$$U = \left\{ q \in \mathbb{R}^n : \ q = q^0 + \sum_{l=1}^{s} \xi_l q^l, \ \xi \in \Xi \right\}. \tag{21}$$

In particular, we consider two uncertainty sets for Ξ:

$$\Xi_1 = \{\xi \in \mathbb{R}^s : \ \|\xi\|_\infty \leq 1\}, \tag{22a}$$
$$\Xi_2 = \{\xi \in \mathbb{R}^s : \ \|\xi\|_2 \leq \rho\}, \tag{22b}$$

which represent, respectively, a *box* and a *ball of radius* ρ. In this section, we present the robust counterparts for the above two sets and show that our formulation mainly results in linear robust counterparts for both sets. However, in Section 5, we present computational results for Ξ_1.

Note that in the model of Section 3.1, only the right-hand side of the MTZ constraints (7–8) is subject to (demand) uncertainty. For such case and the case where the left-hand side of each constraint contains only *one* coefficient of uncertainty, Sungur *et al.* [27] prove that the BN-robust counterpart can be obtained simply by substituting q_j $(j = 1 \ldots n)$ with

$$q_j^0 + \textstyle\sum_{l=1}^{s} |q_j^l|, \tag{23a}$$
$$q_j^0 + \rho\sqrt{\textstyle\sum_{l=1}^{s} (q_j^l)^2}, \tag{23b}$$

for Ξ_1 (22a) and Ξ_2 (22b), respectively. Therefore, the BN-robust counterpart of (7–8) retains the same structure, since only the right-hand side changes.

On the other hand, this is not true for all the inequalities presented in Sections 3.2–3.4. In fact, while the box uncertainty set (22a) always retains linearity, the ball uncertainty set (22b) may lead to *conic quadratic* inequalities when demand uncertainty is not restricted to the right-hand side of the constraints. In what follows, we only present the linear counterparts, since we do not intend to solve Mixed Integer *Non Linear* Programs (MINLP). Unfortunately, this implies that for some uncertainty sets we will not be able to use all the (strengthening) inequalities presented in the previous section for the deterministic model.

First, we consider the *capacity* inequalities (10). The BN-robust counterpart corresponding to the box uncertainty set (22a) is the inequalities:

$$\sum_{i \in V_c} \sum_{a \in \delta^+(i)} q_i^0 x_a^k + \sum_{i \in V_c} \sum_{a \in \delta^+(i)} \sum_{l=1}^{s} |q_i^l| x_a^k \leq Q_k, \ k \in K, \tag{24}$$

whereas the BN-robust counterpart corresponding to the ball uncertainty set (22b) is a set of conic quadratic inequalities, which we do not consider here.

Second, we consider the subtour elimination inequalities (11). Here, only the right-hand side is subject to uncertainty. To construct the BN-robust counterpart it suffices to substitute q_j with (23a) for Ξ_1 and (23b) for Ξ_2, respectively.

Third, we consider the lifted inequalities (13), which lead to conic quadratic inequalities for the ball uncertainty set (22b), whereas for the box uncertainty set (22a) the BN-robust counterpart is:

$$-u_j + u_i + Q_m \sum_{k\in K} x_{ij}^k + (Q_m - q_j^0 - q_i^0) \sum_{k\in K} x_{ji}^k \\ + \sum_{l=1}^{s} \left| (-q_j^l - q_i^l) \sum_{k\in K} x_{ji}^k + q_j^l \right| \leq Q_m - q_j^0 \ , \ (i,j) \in A_c \quad (25)$$

Finally, we consider the *RLT* inequalities of Section (3.4). These always retain linearity since there is only one uncertain (demand) parameter in each inequality, either in the right-hand side or in the left-hand side. So the BN-robust counterpart for Ξ_1 (22a) and Ξ_2 (22b) can again be obtained by substituting q_j with (23a) and (23b), respectively.

4.2 Bertsimas & Sim Robust Model

The robust counterpart developed by Bertsimas & Sim (BS) has two main features: It contains in each constraint a parameter Γ (the protection level) that controls the *degree of conservatism* of the robust solution; it is computationally tractable if the original problem is tractable. Regarding tractability, Bertsimas & Sim give a compact robust counterpart of a given nominal model by introducing a polynomial number of new variables and constraints. We will apply such a approach and use the (strengthening) inequalities presented in Sections 3.2–3.4.

According to BS-model of uncertainty set U, the uncertain demand vector q takes value of the interval $[q^0 - \hat{q}, q^0 + \hat{q}]$, symmetric around the nominal value q^0. The parameter Γ mentioned above denotes the maximum number of coefficients that are allowed to change simultaneously with respect to their nominal values in each constraint. In particular, at most $\lfloor \Gamma \rfloor$ q_is will change to their bounds $\hat{q_j}$s and one will change by $(\Gamma - \lfloor \Gamma \rfloor)$ portion of its bound.

Note that for the MTZ constraints (7–8), there is only one demand parameter in each constraint. Hence, the BS-robust counterpart can be simply obtained by substituting q_j with the quantity $q_j^0 + \Gamma \hat{q}_j$, where $0 \leq \Gamma \leq 1$.

First, consider the capacity inequalities (10). To construct the BS-robust counterpart we denote, for each given $k \in K$, by $\Psi^k \subseteq V_c$ the subset corresponding to those coefficients q_i that are subject to uncertainty and by Γ^k the control parameter for the constraint. Following Bertsimas & Sim construction, we obtain the following BS-robust counterpart with additional variables p_i^k and π^k:

$$\sum_{i \in V_c} q_i^0 \sum_{a \in \delta^+(i)} x_a^k + \sum_{i \in \Psi^k} p_i^k + \Gamma^k \pi^k \leq Q_k,\ k \in K \tag{26a}$$

$$\pi^k + p_i^k \geq \hat{q}_i \sum_{a \in \delta^+(i)} x_a^k,\ i \in \Psi^k, k \in K \tag{26b}$$

$$\pi^k \geq 0,\ k \in K \tag{26c}$$

$$p_i^k \geq 0,\ i \in \Psi^k, k \in K. \tag{26d}$$

Second, consider the subtour elimination inequalities (11), where the uncertainty only appears on the right-hand side of the constraints. For the constraint corresponding to $S \subseteq V_c$, denote by Ψ^S the subset of V_c that corresponds to those q_is that are subject to uncertainty and Γ^S the control parameter for the constraint. Clearly, in this case, we can simply sort $\hat{q}_i$ in non-increasing order and choose the first Γ^S demands where $\lfloor \Gamma^S \rfloor$ can change up to their bounds and the last of the selected demands can only change by $(\Gamma^S - \lfloor \Gamma^S \rfloor)$ portion of its bound.

Third, for the lifted inequalities (13), the BS construction is similar to the one used for the capacity inequalities (10) (see 26a–26d).

Finally, in each of the RLT inequalities of Section 3.4, there is at most one demand coefficient. Hence, the BS-robust counterpart can be obtained by simply substituting q_j with the quantity $q_j^0 + \Gamma \hat{q}_j$.

After setting up the robust counterparts, we need to define the parameter Γ for each constraint. On the one hand, Γ controls the degree of conservatism of the robust solution, that is *guaranteed* to be feasible *up to* Γ simultaneous changes of the coefficients of a given constraint. On the other hand, Bertsimas & Sim also introduce *probability bounds* depending on the value of Γ and representing the *probability of violation* of a constraint if more than Γ coefficients change at the same time. They show that the larger the number of uncertain coefficients in a constraint, the more accurate are the bounds. However, since in many of our inequalities only a few uncertainty coefficients appear, these bounds are not very helpful for deciding the value of Γ. For this reason, we give the following two propositions that allow us to calculate *exactly* the value of Γ corresponding to a given probability of violation for two specific types of constraints. Note that the concept of probability of violation for a given constraint is strictly related to the chance-constrained models that we will present in the next subsection.

Proposition 4 applies when *only one* uncertainty coefficient is present as, for example, in the MTZ constraints(7–8).

Proposition 4 *If q_j ($j \in V_c$) is a uniformly distributed random variable in $[q_j^0 - \hat{q}_j, q_j^0 + \hat{q}_j]$, then any constraint with q_j the only uncertainty coefficient has a probability α of violation for $\Gamma = 1 - 2\alpha$.*

Proof. Since q_j follows a uniform distribution, we can easily calculate the corresponding cumulative distribution function. Hence, by setting $q_j^* = q_j^0 + \hat{q}_j(1 - 2\alpha)$, we can guarantee that $\Pr[q_j \leq q_j^*] \leq \alpha$. Therefore, $\Gamma = 1 - 2\alpha$ provides the desired probability of violation. □

Remark. The above proposition also applies to inequalities (19b) as well as to (19a) since $1 - x_{ij} - x_{ji} = 0$ or 1 due to the integrality condition.

Proposition 5 applies to subtour elimination inequalities (11).

Proposition 5 *Given any $S \subset V_c$, if q_j for any $j \in V_c$ is an independently and symmetrically distributed random variable in $[q_j^0 - \hat{q}_j, q_j^0 + \hat{q}_j]$ with cumulative distribution function $\mathcal{F}_j$ and joint distribution $\mathcal{F}_{q(S)}$, then*

$$\Pr\left[X(S:\bar{S}) \geq \lceil q(S)/Q_k \rceil\right] \geq 1 - \alpha,\ \ k \in K,$$

for Γ computed as follows:

$$\min \Gamma \tag{27a}$$

$$s.t. \quad \textstyle\sum_{i \in S} \xi_i \leq \Gamma \tag{27b}$$

$$\textstyle\sum_{i \in S} \hat{q}_i \xi_i = \mathcal{F}^{-1}_{q(S)}(1 - \alpha) \tag{27c}$$

$$0 \leq \xi_i \leq 1,\ \ i \in S. \tag{27d}$$

Proof. Since the inverse joint distribution function $\mathcal{F}^{-1}_{q(S)}(1 - \alpha)$ can be easily calculated for some classes of distribution functions (e.g., Normal), the LP (27a–27d) selects the uncertainty coefficients such that the sum of their deviations gives the desired value and Γ is minimized. □

4.3 Chance-Constrained Model

In a chance-constrained model, constraints are required to be satisfied with some big probability. We start with the MTZ constraints (7–8), which have the chance-constrained counterpart is follows:

$$\Pr\left[u_j - u_i - Q_m \sum_{k \in K} x_a^k + Q_m \geq q_j\right] \geq 1 - \alpha, \quad a = (i,j) \in A_c, \tag{28a}$$

$$\Pr\left[q_i \leq u_i \leq \sum_{k \in K} Q_k \sum_{a \in \delta^+(i)} x_a^k\right] \geq 1 - \alpha, \quad i \in V_c, \tag{28b}$$

which mean that the constraints can be violated with probability at most α. In particular, given a cumulative distribution $\mathcal{F}_j$ for the demand parameter q_j, the above are equivalent to:

$$u_j - u_i - Q_m \sum_{k \in K} x_a^k + Q_m \geq \mathcal{F}_j^{-1}(1 - \alpha), \quad a = (i,j) \in A_c, \tag{29a}$$

$$\mathcal{F}_j^{-1}(1 - \alpha) \leq u_i \leq \sum_{k \in K} Q_k \sum_{a \in \delta^+(i)} x_a^k, \quad i \in V_c. \tag{29b}$$

Note that the chance-constrained counterpart (29a–29b) retains linear.

The chance-constrained counterpart of the capacity inequalities (10) involves *nonlinear* constraints. To ease notation we let $x_i^k = \sum_{a\in\delta^+(i)} x_a^k$ and let x^k denote the (column) vector $(x_i^k : i \in V_c)$. The chance-constrained counterpart can be written as follows:

$$\Pr[q^T x^k \leq Q_k] \geq 1-\alpha,\ k \in K. \tag{30}$$

When q follows a normal distribution $\mathcal{N}(\mu, \Lambda)$ with mean (vector) μ and covariance (matrix) Λ, the above chance constraint can be reformulated as the following second order cone constraint:

$$\sqrt{(x^k)^T \Lambda x^k} \leq \frac{Q_k - \mu^T x^k}{\Phi^{-1}(1-\alpha)},\ k \in K, \tag{31}$$

where Φ is the cumulative distribution function of the standard normal distribution. Here we are interested in the case where demands are not correlated (i.e., $\lambda_{ij} = 0,\ i \neq j \in V_c$). So we can rewrite (31) in the following way:

$$\mu^T x^k + \Phi^{-1}(1-\alpha)\sqrt{\sum_{i\in V_c} \lambda_i^2 (x_i^k)^2} \leq Q_k,\ k \in K. \tag{32}$$

To obtain a linear formulation we can substitute the non-linear term on the left-hand side with the linear over-estimator $\Phi^{-1}(1-\alpha)\sum_{i\in V_c} \lambda_i x_i^k$, obtaining an approximated (linear) chance constraint (λ_i is the demand standard deviation for costumer i).

Next let us consider the chance-constrained counterpart of the subtour elimination inequalities (11), which is as follows:

$$\Pr\left[X(S:\bar{S}) \geq \lceil q(S)/Q_k \rceil\right] \geq 1-\alpha,\ k \in K,\ S \subseteq V_c. \tag{33}$$

If $\mathcal{F}_{q(S)}$ is the joint distribution function of the random variables $q_i,\ i \in S$, then the above is equivalent to:

$$X(S:\bar{S}) \geq \left\lceil \mathcal{F}_{q(S)}^{-1}(1-\alpha)/Q_k \right\rceil,\ k \in K,\ S \subseteq V_c, \tag{34}$$

where $\mathcal{F}_{q(S)}^{-1}(1-\alpha)$ can be calculated for some classes of distribution functions (e.g., *Normal*), when demands are independently distributed and follow the same distribution with different parameters. For example, when $q(S) \sim \mathcal{N}(\mu_S, \Lambda)$, where $\mu_S = \sum_{i\in S} \mu_i$ is the sum of the means and Λ is the covariance matrix, then we have a tractable case and (34) can be replaced by

$$X(S:\bar{S}) \geq \lceil q^*(S)/Q_k \rceil,\ k \in K,\ S \subseteq V_c,$$

where $q^*(S)$ is calculated as follows:

$$\Pr\left[q(S) \geq q^*(S)\right] = \Pr\left[\frac{q(S)-\mu_S}{\sqrt{|\Lambda|}} \geq \frac{q^*(S)-\mu_S}{\sqrt{|\Lambda|}}\right],$$

and

$$q^*(S) = \mu_S + \Phi^{-1}(1-\alpha)\sqrt{|\Lambda|}.$$

The chance-constrained counterpart of the lifted inequalities (13) also involves nonlinear constraints, which we do not consider here.

Finally, the chance-constraint counterpart of the *RLT* inequalities of Section 3.4 retains linearity, since there is only one random variable which appears as a coefficient of one or more decision variables. In this case we can apply the same idea used for the MTZ constraints. For example, considering the chance-constrained counterpart of the RLT inequalities (19b), we get

$$\Pr\left[u_j \geq z_{ij} + y_{ji} + q_j(1 - x_{ij} - x_{ji})\right] \geq 1 - \alpha,$$

which is equivalent to

$$u_j \geq z_{ij} + y_{ji} + \mathcal{F}_j^{-1}(1-\alpha)(1 - x_{ij} - x_{ji}).$$

5 Computational Experiment

In Section 5.1, we present percentage gaps for the lower bounds corresponding to the LP relaxation of different formulations for the deterministic model. In Section 5.2, we present three performance measures, by which we analyze the solutions of the three uncertainty models considered in Section 4 (i.e., BN, BS and CC).

Our computational experiments use two sets of benchmark instances: Golden et al. [16] and Prins & Prodhon (`http://prodhonc.free.fr/`), which are denoted by G and P, respectively. G instances correspond to *single-depot* HVRP with *unlimited fleet size* and fixed costs. P instances were originally generated for the *homogeneous location routing problem*, so we modify them to obtain *multi-depot* HVRP with *limited fleet size*. In particular, according to the solutions presented in `http://prodhonc.free.fr/`, we limit the number of vehicles to that needed to serve the customers. We change the capacity of vehicles to define a heterogenous fleet (Q_k). We assign a coefficient (OC_k) as operational (traveling) cost for each type, so that the matrix c_a^k is calculated by taking the distance between nodes and multiplying it by OC_k. In Table 1 we report for each instance of type P, the number of vehicles (NO. Veh.), the original capacity (Cap.) and for each type ($k = 1 \ldots 5$) the corresponding operational cost (OC_k) and capacity (Q_k).

5.1 Lower Bounds for the Deterministic Model

Table 2 shows percentage gaps between the lower bounds and the upper bounds for different formulations of the deterministic HVRP. The lower bounds are obtained by relaxing the integrality conditions and the upper bounds are obtained from Yaman [29]. The single-depot HVRP with fixed cost is considered. We do not claim that these bounds are the best known bounds. Instead we

Table 1 Vehicle type details

Instance	NO. Veh.	Cap.		$k=1$	2	3	4	5
P-20-5-5-1a	5	70	OC_k	1	1.2	1.4	1.6	2
			Q_k	70	100	130	160	190
P-20-5-3-1b	3	150	OC_k	1	1.2	1.4		
			Q_k	150	200	250		
P-20-5-5-2a	5	70	OC_k	1	1.2	1.4	1.6	2
			Q_k	70	100	130	160	190
P-20-5-3-2b	3	150	OC_k	1	1.2	1.4		
			Q_k	150	200	250		

Table 2 Gap on percentage for the deterministic models

Instance	MTZ	Cap.	DL	RLT	RLT^M
G-n20-k5	76.78	13.46	11.30	11.16	9.93
G-n20-k3	96.56	3.50	3.17	3.16	3.04
G-n20-k5	77.84	18.09	17.09	16.96	13.25
G-n20-k3	96.60	5.01	4.81	4.78	4.27
G-n50-k6	84.75	9.55	9.03	9.01	7.88
G-n50-k3	95.99	5.91	5.74	5.73	5.51
G-n50-k3	84.87	13.72	12.85	12.79	11.06
G-n50-k3	85.70	10.06	9.18	9.15	7.39
G-n75-k4	71.95	12.18	9.80	9.79	8.17
G-n75-k6	79.55	15.30	13.69	13.68	12.74
G-n100-k3	93.31	6.53	6.06	6.06	5.59
G-n100-k3	85.53	12.94	12.09	12.06	10.35

would like to compare the performance of the RLT approach presented in Section 3.4 and the lifting approach presented in Section 3.3. The first column represents the instances. For example, `G-n20-k5` has 20 vertices, 5 types of vehicles and unlimited number of vehicles of each type. The second column (MTZ) corresponds to the LP relaxation of the standard MTZ formulation (1–9). The third column (Cap.) corresponds to the LP relaxation of the standard MTZ formulation after adding the capacity inequalities (10). The fourth column (DL) is obtained by substituting (7) and (8) with (13) and (14) in (1–9). The fifth column (RLT) is obtained by replacing (7) and (8) with (17)–(20b). The big-M method can be used to linearize the nonlinear term in the RLT (16) as follows. The gap for the RLT^M is provided in its corresponding column.

$$y_{ij} \leq u_i, \ i,j \in V_c, \tag{35a}$$
$$y_{ij} \geq u_i - M(1 - \textstyle\sum_{k\in K} x_{ij}^k), \ i,j \in V_c, \tag{35b}$$
$$y_{ij} \leq M \textstyle\sum_{k\in K} x_{ij}^k, \ i,j \in V_c, \tag{35c}$$

5.2 Experiments with Demand Uncertainty

We start with describing how the data uncertainty is constructed, then we explain the performance measures used and finally we analyze the computational results.

Uncertain Data To build demand uncertainty sets for the BS and BN robust models, we allow q_i to vary up to a fixed percentage of its nominal value so that $q_i \in [q_i^0 - \upsilon q_i^0,\ q_i^0 + \upsilon q_i^0]$, where q_i^0 is the demand nominal value and $\upsilon = 0.1$ or 0.2. To build uncertainty sets for the CC model, it is quite common to consider a normal distribution based on the mean and the variance calculated for a sample. Hence, we assume that the demand of each customer follows the normal distribution $\mathcal{N}(\mu_i, \lambda_i^2)$ with $\mu_i = q_i^0$, and $\lambda_i^2 = \frac{0.16}{12} q_i^0$. Notice that we set the variance equal to the variance of the uniform distribution that we calculated for the RO cases. In this case, 91% of the interval defined previously is covered by the normal distribution function.

Performance Measures We compare our solutions according to three performance measures.

First, we compute the *extra cost* E^a required to pay for achieving a certain level of validity of routes:

$$\mathsf{E}^a := \frac{z^a - z^{det}}{z^a} \times 100,$$

where z^a denotes the optimal value of the uncertain model (a can be bs, bn and cc for BS, BN and CC models, respectively) and z^{det} is the optimal value for the deterministic case.

In case of failure, there are two possible strategies. On the one hand, one may assume that vehicles return to the depot and do not resume the interrupted (failed) route, so the remaining customers on the failed route are left unserved. This is known as *allowed lost sales* (ALS). The second performance measure represents the *number of unmet customers* (and the corresponding unmet demand). On the other hand, if lost sale is not allowed (NALS), the vehicle returns to the depot for a replenishment and then resumes the route starting from the first customer who was left unserved. The third performance measure calculates the *recourse cost.*

Since the probability of failure (*risk level*) and the cost are conflicting goals, we would like to find a proper threshold. Risk level is an important parameter in CC and BS models (denoted by α in Sections 4.2 and 4.3) by which we can adjust the conservativeness of the solutions. From the sensitivity analysis for MIP, we know that for small perturbations of parameters, the optimal solution may remain unchanged and from some point, the optimal solution will change. However, the behavior of the optimal solution with respect to changes in MIP parameters is not quite predictable and the value function in MIP is in general non-convex. This topic has been widely studied (see [7]). By changing the risk level we can measure the sensitivity of the optimal solution and find the thresholds at which the optimal solution will change. In many cases of MIP, a full description of the convex hall of the feasible region is not available and constraints may not define facets of the convex hull, hence changing the parameters may affect neither the optimal solution nor the objective value. On the other hand, if an optimal solution is cut off as a result of varying parameters, the effect can be dramatic from changing the optimal solution to infeasible solution. Therefore, particularly in practice when resources are limited, it is vital to define appropriate risk levels so that not only the solution is feasible but also unnecessary extra costs are not imposed. One way of identifying the threshold is to define different scenarios for the risk level. Here in addition to the nominal case which represents $\alpha_0 = 0.5$, we consider 9 scenarios for the risk level ($\alpha_1 = 0.40,\ \alpha_2 = 0.30,\ \alpha_3 = 0.25,\ \alpha_4 = 0.20, \alpha_5 = 0.10,\ \alpha_6 = 0.05,\ \alpha_7 = 0.03,\ \alpha_8 = 0.01,\ \alpha_9 = 0.001$). Note that the larger the risk level, the higher the probability of violating a constraint. We solve the CCP and BS-RO deterministic counterparts of the instances for all these scenarios and calculate the aforementioned performance measures for each scenario. As formulated in the previous section, the protection level of the BS-RO (Γ) is calculated for each risk level. Then, among the risk level scenarios, the optimal one can be suggested.

Computational results In this experiment, we consider the variable routing cost for the data sets. All experiments are carried out on a Dell Precision T1600 computer with a 3.4 GHz Intel Xeon Processor and 16 GB RAM running Ubuntu Linux 12. Also note that we use our B&C method for the nominal problem and the BN-RO and the default CPLEX solver for the BS-RO. When a user defined B&C method is run in CPLEX, by default CPLEX uses only one thread.

Tables 3 and 4 present E^{bs} and E^{bn} values with $\upsilon = 10\%$ and $\upsilon = 20\%$, respectively. Table 5 presents E^{cc} values with $\upsilon = 20\%$. All running times are in seconds. Note that, when the BS optimal value equals the BN optimal value, we do not need to run other risk levels since they will give the same results. When this happens, we use bold numbers in Tables 3 and 4 for the corresponding percentage of extra cost.

In order to calculate for the second and third performance measures, we generate random demands for all customers from their defined distribution functions to simulate the actual situations. Table 6 reports the results for the average of the second and third performance measures for 100 simulations with $\upsilon = 20\%$. For each instance, we use abbreviations as follows: U (Unmet Demands), N (Number of Unmet demands), R (Recourse Cost) and NR (Number of Routes). As the numerical result suggests, we do not need to set a very low risk level to achieve 100% valid routes.

Figure 1 illustrates the actual costs, the optimal costs of the BS-RO for the defined scenarios of the risk level and also the optimal cost of the BN-RO. The actual cost is calculated based on the BS-RO solution for each scenario as follows. For each scenario of the risk level, the routes are set according to its BS-RO solution and then customer demands are generated from their pre-defined intervals, assuming that they follow a uniform distribution (100 realizations for each demand). Since in only one scenario the risk level is zero, there is a possibility of failure for the other scenarios. First, we assume that no lost sale is allowed, so the recourse actions are performed to serve unmet customers and the related recourse costs are calculated and added to the optimal costs obtained by the BS-RO. We call this total cost as the *actual cost with recourse action*. Figure 1 illustrates these three cost graphs for instance G-n20-k5. One can observe that if the risk level is set at a big value ($\alpha = 0.40$), the actual cost is even less than when the risk level is very small ($\alpha \leq 0.10$). It suggests that the extra cost paid to prevent the route validity for certain level is not necessary. In this specific problem, if the risk level is set to $\alpha = 0.20$, the total cost will be minimum. We can conclude that a lower risk level does not necessarily lead to a better result. Some unnecessary costs may be imposed without any significant outcome for the system. Figure 1 and Table 8 (NALS) provide the optimal level of risk level for each problem of the BS-RO and the CCP. Obviously, the BN-RO is too conservative and imposes unnecessary costs.

On the other hand, we can assume that lost sales/unmet customers are allowed in some cost. This means when a failure occurs the vehicle returns to the depot and does not resume the route, so the remaining customers on the failed route will be left unserved. To identify the optimal risk level in this case, let us assume a simple case where each lost sale has the same cost of f. We undertake a pairwise comparison among different scenarios to find out under which condition one is better than the other. Let C_1, C_2, n_1 and n_2 be the optimal cost and the number of unmet customers for two risk scenarios 1 and 2, respectively. If $(n_1 - n_2)f \leq C_2 - C_1$, then scenario 1 is better than scenario 2. Otherwise, scenario 2 is better than scenario 1. Therefore, a risk level can be the best scenario for a specific range of lost sale costs.

Table 9 presents intervals for f in which a risk level is the optimal when lost sales is allowed (ALS). For instance, for Instance G-n20-k5, for the BS-RO when $f \in [0,\ 19.97]$ and $f \in [19.97,\ 40.23]$, then the best risk levels are $\alpha = 0.4$ and $\alpha = 0.3$, respectively. However, $\alpha = 0.25$ cannot be the optimal risk level as it has the same cost of $\alpha = 0.2$ while there are unmet customers. So, when $f \in [40.23,\ \infty)$ $\alpha = 0.2$ is the optimal risk level. As this analysis also suggests, the smaller risk levels are not necessarily the best options.

6 Conclusions

Robust optimization and chance constrained programming are important tools in the presence of data uncertainty. We have studied the HVRP under demand uncertainty. Rather than choosing a particular uncertainty model, we have considered three well-known models and analyzed them using different performance measure.

The notion of risk level allows us to undertake insightful comparisons. From a computational point of view, a potential topic for future research is to improve the branch-and-bound algorithm in order to consider larger instances. From a theoretical point of view, it would be interesting to see if the concept of risk level can be addressed more systematically, with possible consideration of bi-objective models and computing *Pareto* solutions.

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Table 3 The deterministic optimal objective value and the first performance measure for BS-RO and BN-RO ($\upsilon = 0.1$), where N indicates unsolved.

Inst.		Nom.	BS							BN
			Γ=1.14	2.34	3.01	3.77	5.74	7.35	8.76	
			α=0.40	0.30	0.25	0.20	0.10	0.05	0.03	
UL. Veh.										
G-n20-k5	E	623.22	1.07	1.33	1.45	1.91	3.19	3.19	**4.30**	4.30
	T		4147	2199	1264	859	2103	1366	4429	2183
G-n20-k3	E	387.18	0.82	1.04	1.04	1.04	1.92	**3.31**	-	3.31
	T		24561	5208	6317	1889	2470	29387		43392
G-n20-k5	E	742.87	N	N	N	N	4.97	4.97	N	6.06
	T						5632	1079		1206
G-n20-k3	E	415.03	0.00	0.00	1.96	2.20	2.35	**2.59**	-	2.59
	T			1967	6755	4168	2528	2364		21450
L. Veh.										
P-20-5-5-1a	E	234.36	0.65	N	N	**0.77**	-	-	-	0.77
	T		30806			11679				37518
P-20-5-3-1b	E	217.58	0.00	0.00	0.00	0.00	**0.55**	-	-	0.55
	T		1354	626	956	733	793			3427
P-20-5-5-2a	E	194.46	0.00	**3.06**	-	-	-	-	-	3.06
	T		1124	1714						2228.71
P-20-5-3-2b	OE	180.48	0.00	0.00	0.00	0.00	**3.99**	-	-	3.99
	T					14	117			417

Table 4 The deterministic optimal objective value and the first performance measure for BS-RO and BN-RO ($\upsilon = 0.2$)

Inst.		Nom.	BS									BN
			Γ=1.14	2.34	3.01	3.77	5.74	7.35	8.76	10.39	16.99	
			α=0.40	0.30	0.25	0.20	0.10	0.05	0.03	0.01	0.001	
UL. Veh.												
G-n20-k5	E	623.22	1.31	3.09	4.12	4.12	6.72	7.23	7.87	8.55	8.55	9.11
	T		2102	1879	3324	1439	915	1225	1187	959	806	1163
G-n20-k3	E	387.18	1.03	1.89	3.20	3.20	4.15	**4.56**	-	-	-	4.56
	T		4355	5025	3269	1245	1269	1409	-			494
G-n20-k5	E	742.87	N	5.49	7.39	N	N	10.35	10.35	10.35	N	11.07
	T			3921	3679			17680	48302	10205		545
G-n20-k3	E	415.03	2.15	2.53	2.5	N	7.62	8.15	8.89	**10.49**	-	10.49
	T		20488	9939	4446		15668	6528	25907	38829		38794
L. Veh.												
P-20-5-5-1a	E	234.36	0.77	0.76	0.76	1.04	**4.65**	-	-	-	-	4.65
	T		20929	4903	-	14132	21130					88165
P-20-5-3-1b	E	217.58	0.00	**0.55**	-	-	-	-	-	-	-	0.55
	T		467	645								122
P-20-5-5-2a	E	194.46	2.97	2.97	2.97	2.97	**4.59**	-	-	-	-	4.59
	T		3114	835	1682	950	1229					
P-20-5-3-2b	E	180.48	0.00	3.84	3.84	3.84	3.84	5.08	**5.73**	-	-	5.73
	T		22	158	156	146	114	64	250			104

Table 5 The deterministic optimal objective value and the first performance measure for CCP ($\upsilon = 0.2$)

		Nom.	CCP								
Inst.			$\alpha = 0.40$	0.30	0.25	0.20	0.10	0.05	0.03	0.01	0.001
UL. Veh.											
G-n20-k5	E	623.22	2.72	6.84	8.80	10.01	18.81	24.79	29.21	34.79	41.73
	T		278	190	69	37	102	40	21	25	6
G-n20-k3	E	387.18	2.56	4.27	5.09	5.16	11.63	14.65	17.51	19.87	28.92
	T		1287	864	402	132	312	172	212	118	38
G-n20-k5	E	742.87	5.87	10.21	14.01	14.01	24.17	32.43	38.08	43.16	58.12
	T		2622	293	2096	264	1674	242	532	53	22
G-n20-k3	E	415.03	2.20	8.52	10.23	11.41	17.75	24.70	27.89	31.09	38.73
	T		509	3435	2472	1044	389	408	547	257	38
L. Veh.											
P-20-5-5-1a	E	234.36	N	N	N	N	N	N	N	N	N
	T										
P-20-5-3-1b	E	217.58	0.00	0.55	0.55	0.55	0.98	6.62	6.62	12.03	13.23
	T		786	408	195	139	162	399	295	1197	694
P-20-5-5-2a	jE	194.46	3.06	3.06	4.81	4.81	7.10	7.10	15.41	18.19	21.95
	T		1722	1702	2357	566	1003	439	4366	7845	1808
P-20-5-3-2b	E	180.48	0.00	3.99	3.99	5.35	10.27	11.44	12.35	12.35	21.67
	T		21	134	139	58	398	411	895	428	4194

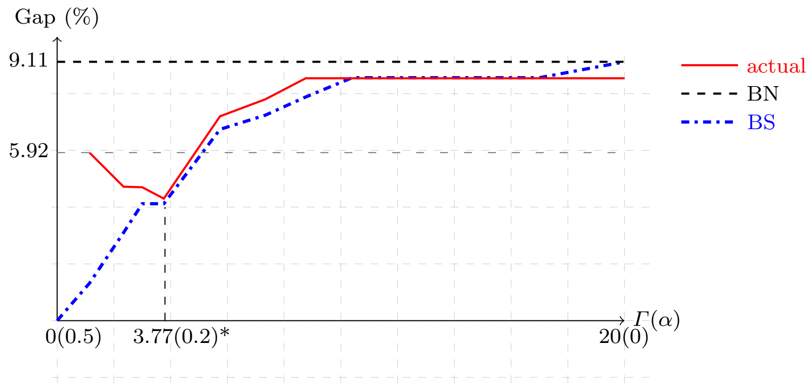

Fig. 1 Risk levels, optimal costs and actual costs for Instance G-n20-k5

Table 6 Second and third performance measures for the BS ($\upsilon = 0.2$)

Inst.	Γ	1.14	2.34	3.01	3.77	5.74	7.35	8.76	10.39	16.99
	α	0.40	0.30	0.25	0.20	0.10	0.05	0.03	0.01	0.001
UL. Veh.										
G-n20-k5	U	18.55	5.93	2.42	0	0	0	0	0	0
	N	8.30	2.5	0.80	0	0	0	0	0	0
	R	28.58	9.55	2.51	0	0	0	0	0	0
G-n20-k3	U	3.93	N	0.25	0	0	0	0	0	0
	N	0.25		0.01	0	0	0	0	0	0
	R	0.97		0.36	0	0	0	0	0	0
G-n20-k5	U	N	3.07	4.31	N	N	0	0	0	0
	N		0.27	0.09			0	0	0	0
	R		12.93	2.40			0	0	0	0
G-n20-k5	U	0.48	0.73	0	0	0	0	0	0	0
	N	0.04	0.03	0	0	0	0	0	0	0
	R	1.32	1.40	0	0	0	0	0	0	0
L. Veh.										
20-5-5-1a	U	N	0	0	0	0	0	0	0	0
	N		0	0	0	0	0	0	0	0
	R		0	0	0	0	0	0	0	0
20-5-3-1b	U	0.65	0	0	0	0	0	0	0	0
	N	0.03	0	0	0	0	0	0	0	0
	R	0.40	0	0	0	0	0	0	0	0
20-5-5-2a	U	0	0	0	0	0	0	0	0	0
	N	0	0	0	0	0	0	0	0	0
	R	0	0	0	0	0	0	0	0	0
20-5-3-2b	U	0.30	0	0	0	0	0	0	0	0
	N	0.02	0	0	0	0	0	0	0	0
	R	0.16	0	0	0	0	0	0	0	0

Table 7 Second and third performance measures for the CCP ($\upsilon = 0.2$)

Inst.	Γ	1.14	2.34	3.01	3.77	5.74	7.35	8.76	10.39	16.99
	α	0.40	0.30	0.25	0.20	0.10	0.05	0.03	0.01	0.001
UL. Veh.										
G-n20-k5	U	46.96	52.55	43.72	43.01	20.62	4.32	1.89	0	0
	N	1.37	1.25	0.99	1.00	0.37	0.09	0.02	0	0
	R	42.88	46.96	31.93	35.41	13.07	3.12	0.59	0	0
G-n20-k3	U	19.12	16.05	13.16	16.99	1.62	1.43	0	0	0
	N	0.75	0.38	0.25	0.31	0.03	0.03	0	0	0
	R	20.51	11.87	7.63	8.9	1	1.33	0	0	0
G-n20-k5	U	49.21	37.07	29.38	25.38	5.48	4.52	1.20	1.19	0
	N	1.64	0.97	0.64	0.55	0.1	0.05	0.02	0.03	0
	R	50.79	39.3	23.29	23.05	4.85	1.89	1.14	1.63	0
G-n20-k5	U	40.8	4.56	6.03	17.46	3.24	0	0	0	0
	N	0.82	0.17	0.20	0.30	0.06	0	0	0	0
	R	34.36	6.91	9.04	9.26	1.90	0	0	0	0
UL. Veh.										
20-5-5-1a	U	N	N	N	N	N	N	N	N	N
	N									
	R									
20-5-3-1b	U	1.05	0	0	0	0	0	0	0	0
	N	0.05	0	0	0	0	0	0	0	0
	R	0.67	0	0	0	0	0	0	0	0
20-5-5-2a	U	0	0	0	0	0	0	0	0	0
	N	0	0	0	0	0	0	0	0	0
	R	0	0	0	0	0	0	0	0	0
20-5-3-2b	U	0.32	0	0	0	0	0	0	0	0
	N	0.02	0	0	0	0	0	0	0	0
	R	0.16	0	0	0	0	0	0	0	0

Table 8 Best scenario for the risk level when lost sales are not allowed for BS and CCP

α	0.40	0.30	0.25	0.20	0.10	0.05	0.03	0.01	0.001	Best Scen.
Inst.					BS					
UL. Veh.										
G-n20-k5	5.92	4.72	4.70	4.30	7.21	7.80	8.55	8.55	8.55	0.20
G-n20-k3	3.55	4.65	3.40	3.31	4.33	4.78	4.78	4.78	4.78	0.20
G-n20-k5	N	7.24	7.72	N	N	10.35	10.35	10.35	N	0.30
G-n20-k3	2.52	2.93	2.59	N	8.25	8.87	8.89	10.49	10.49	0.40
L. Veh.										
20-5-5-1a	0.77	0.77	0.77	1.05	4.88	4.88	4.88	4.88	4.88	0.25
20-5-3-1b	0.30	0.55	0.55	0.55	0.55	0.55	0.55	0.55	0.55	0.40
20-5-5-2a	3.06	3.06	3.06	3.06	4.81	4.81	4.81	4.81	4.81	0.25
20-5-3-2b	0.09	3.99	3.99	3.99	3.99	5.35	6.08	6.08	6.08	0.40
					CCP					
UL. Veh.										
G-n20-k5	0.10	0.14	0.14	0.16	0.21	0.25	0.29	0.35	0.42	0.40
G-n20-k3	0.08	0.07	0.07	0.07	0.12	0.15	0.18	0.20	0.29	0.20
G-n20-k5	0.13	0.16	0.17	0.17	0.25	0.33	0.38	0.43	0.58	0.40
G-n20-k3	0.10	0.10	0.12	0.14	0.18	0.25	0.28	0.31	0.39	0.30
L. Veh.										
20-5-5-1a	N	N	N	N	N	N	N	N	N	N
20-5-3-1b	0.00	0.01	0.01	0.01	0.01	0.07	0.07	0.12	0.13	0.40
20-5-5-2a	0.03	0.03	0.05	0.05	0.07	0.07	0.15	0.18	0.22	0.30
20-5-3-2b	0.00	0.04	0.04	0.05	0.10	0.11	0.12	0.12	0.22	0.40

Table 9 Best intervals of the lost sale cost for each scenario for the BS and the CCP

	α=0.4	0.3	0.25	0.2	0.1	0.05	0.03	0.01	0.001
					BS				
UL. Veh.									
G-n20-k5	[0,19.97]	[19.97,40.23]	-	[40.23,∞)	-	-	-	-	-
G-n20-k3	[0,36.57]	-	[36.57,∞)	-	-	-	-	-	-
G-n20-k5	N	[0,78.22]	N	N	-	-	-	-	-
G-n20-k3	[0,163.2]	-	[163.2,∞)		-	-	-	-	-
L. Veh.									
20-5-5-1a	[0,∞)	-	-	-	-	-	-	-	-
20-5-3-1b	[0,39.8]	[39.8,∞)	-	-	-	-	-	-	-
20-5-5-2a	[0,∞)	-	-		-	-	-	-	-
20-5-3-2b	[0,360.3]	[360.3,∞)	-	-	-	-	-	-	-
					CCP				
UL. Veh.									
G-n20-k5	[0,100.25]	-	-	-	[100.25,133.19]	[133.19, 392.81]	[392.81,1739.85]	[1739.85,∞)	-
G-n20-k3	[0,17.84]	[17.84, 24.60]	-	[24.6,115.11]	-	[115.11, 758.1]	[758.1, ∞)	-	-
G-n20-k5	[0, 48.11]	[48.11,85.64]	-	[85.64, 167.61]	[167.61,1227.3]	[12273,1399]	-	[1399, 7445]	[1399,∞)
G-n20-k3	[0, 40.36]	[40.36, 348.05]	-	-	[348.05, 481.01]	[481.01,∞)	-	-	-
L. Veh.									
20-5-5-1a	N	-	-	-	-	-	-	-	-
20-5-3-1b	[0,23.88]	[23.88,∞)	-	-	-	-	-	-	-
20-5-5-2a	[0,∞)	-	-	-	-	-	-	-	-
20-5-3-2b	[0,360.3]	[360.3, ∞)	-	-	-	-	-	-	-